The Heisenberg group and $\mathrm{SL}_2(\mathbb{R})$

a survival pack

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Phase space, quantum blobs

and squeezed coherent states

- *Quantum blobs*[1] are the smallest phase space units of phase space compatible with the uncertainty principle of quantum mechanics;
- Quantum blobs are in a bijective correspondence with the *squeezed coherent states*[2] from standard quantum mechanics, of which they are a phase space picture;
- Quantum blobs have the *symplectic group* as group of symmetries;
- Thus: blobs (aka squeezed states) are a family invariant under quadratic Hamiltonians, which transform them *geomentrically*, that is by a change of variables.

Question: Do we have other states transformed geometrically by quadratic Hamiltonians?

Answer (well-known, yet will be revised): For the harmonic oscillator any state is moving geometrically in the phase space.

[1] M. A. d. Gosson, "Quantum Blobs", 2013.

[2] Gazeau, *Coherent States in Quantum Physics*, 2009; M. d. Gosson, *Symplectic Geometry and Quantum Mechanics*, 2006.

Ladder Operators

and the Hermite functions

Let P and Q satisfy CCR: $[Q, P] = i\hbar I$ of the Weyl algebra $\mathfrak{h}_1$. Consider complexification of $\mathfrak{h}_1$ and define operators:

$$a^{\pm} = \sqrt{\frac{m\omega}{2\hbar}} \left(Q \mp \frac{i}{m\omega} P \right), \quad \text{then} \quad [a^-, a^+] = 1. \tag{1}$$

For a solution $|0\rangle$ of $a^- |0\rangle = 0$ we define $|n\rangle = (\pi^n n!)^{-1/2} (a^+)^n |0\rangle$, then:

1. $(a^-)^* = a^+$ on $L_2(\mathbb{R})$.
2. The name *ladder* operators is explained by the diagram:

   $$|0\rangle \underset{a^-}{\overset{a^+}{\rightleftarrows}} |1\rangle \underset{a^-}{\overset{a^+}{\rightleftarrows}} |2\rangle \underset{a^-}{\overset{a^+}{\rightleftarrows}} |3\rangle \underset{a^-}{\overset{a^+}{\rightleftarrows}} \ldots$$

   Since
   $(a^-) |n\rangle = -(\pi n)^{1/2} |n-1\rangle, \quad (a^+) |n\rangle = (\pi(n+1))^{1/2} |n+1\rangle$
3. Orthonormality: $\langle n|k\rangle = \delta_{nk}$.

Ladder Operators

and representation theory

The above construction relays on CCR: $[Q, P] = i\hbar I$ or $[a^-, a^+] = 1$ only. No specific realisation is assumed. Conclusions:

- There are different vacuums $|0\rangle$ (that is $a^- |0\rangle = 0$) for different $m\omega$.
- Any normalised vacuum $|0\rangle$ creates the orthonormal basis $\{|n\rangle\}_n$ of an irreducible invariant space.
- Any two irreducible spaces are isomorphic by $|n\rangle \to |n\rangle'$.
- Thus, it provides the (constructive!) proof of the Stone-von Neumann theorem on the uniqueness of representation of CCG.

Remark 1.

- There is no genuinely "non-squeezed" states, they are all squeezed in a different way.
- All vacuums are minimising uncertainty $\Delta Q \cdot \Delta P (\geqslant \frac{\hbar}{2})$.
- Obviously, this approach inspired Bargmann to produce classification of UIRs of $SL_2(\mathbb{R})$.

Ladder Operators

and quantum harmonic oscillator

Hamiltonian of the harmonic oscillator.

$$H = \frac{\hbar\omega}{2}(a^+a^- + a^-a^+) = \hbar\omega(a^+a^- + \tfrac{1}{2}) = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2. \qquad (2)$$

Using identities in 2 we obtain spectral decomposition

$$H\left|n\right\rangle = \hbar\omega(n + \tfrac{1}{2})\left|n\right\rangle.$$

1. The spectrum of the harmonic oscillator is discrete.
2. The eigenfunctions are provided by the $\left|n\right\rangle$.
3. The ladder operators acts on the spectrum by a shift $\hbar\omega$ due to the commutation relation $[H, a^\pm] = 2a^\pm$:

$$\begin{aligned} H(a^+\left|k\right\rangle) &= (a^+H + 2a^+)\left|k\right\rangle = a^+(H\left|k\right\rangle) + 2a^+\left|k\right\rangle \\ &= (2k+1)a^+\left|k\right\rangle + 2a^+\left|k\right\rangle = (2k+3)a^+\left|k\right\rangle. \end{aligned}$$

Hamiltonian from Ladder Operators

pros and contras

- Representation independent.
- Inspired ladder technique for other Hamiltonians, e.g. hydrogen atom by Schrödinger or resent works in SUSY QM.
- Time evolution of an arbitrary superposition $\sum a_n |n\rangle$ is $\sum e^{-i\omega(n+1/2)t} a_n |n\rangle$.
- In configuration space (the Schrödinger model) the dynamic is a Gauss-type (quadratic Fourier) integral transform.
- There is a specific space—Fock–Segal–Bargmann (FSB) space—which makes the dynamic geometric.

Harmonic oscillator

in FSB representation

Using the displacement operator $D(z) = e^{(\bar{z}a^+ + za^-)/2} = e^{xQ+yP}$, $z = m\omega x + iy$ (the representation of the Heisenberg group, in fact) we create the coherent states $|z\rangle = D(z)\,|0\rangle$ and the coherent state transform:

$$\mathcal{W}: \ f \mapsto f(z) = \langle f | z \rangle \tag{3}$$

The image—the Fock–Segal–Bargmann space—consists of analytic functions on $\mathbb{C}$. Hamiltonian of the harmonic oscillator:

$$\begin{aligned} H &= \frac{1}{2m}(d\tilde{\sigma}_h^x)^2 + \frac{m\omega^2}{2}(d\tilde{\sigma}_h^y)^2 \\ &= \frac{1}{2m}\partial_{xx}^2 + \frac{m\omega^2}{2}\partial_{yy}^2 \\ &\quad + i\pi h\left(m\omega^2 x\partial_y - \tfrac{1}{m}y\partial_x\right) - \pi^2 h^2\left(\tfrac{m\omega^2}{2}x^2 + \tfrac{1}{2m}y^2\right). \end{aligned} \tag{4}$$

The oscillator's dynamics in FSB space is geometric rotation: $A_t : f(z) \mapsto e^{-\pi\hbar(i\omega t - m\omega y^2 - x^2/(m\omega))} f(e^{-2\pi i\hbar\omega t} z)$ despite of the presence of the second derivatives!

Harmonic oscillator

a solution in FSB space

The mystery resolved:[3]
Functions in FSB transform are analytic functions of the variable $z = m\omega x + iy$, thus the second order "Laplacian"

$$\frac{1}{2m}\partial^2_{xx} + \frac{m\omega^2}{2}\partial^2_{yy}$$

vanishes on FSB space.
The key element:
the representation on the phase space is reducible, giving the room for an additional condition(s), e.g. analyticity, to specify vectors from the irreducible component.
Question: Are their other examples of a geometric dynamic?

[3] Almalki and Kisil, "Geometric Dynamics of a Harmonic Oscillator, Non-Admissible Mother Wavelets and Squeezed States", 2018.

The shear Lie algebra

just one nilpotency step away from the Heisenberg–Weyl

Let $\mathfrak{a}$ be the three-step nilpotent Lie algebra whose basic elements $\{\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3, \mathsf{X}_4\}$ with the following non-vanishing commutators[4]:[5]

$$[\mathsf{X}_1, \mathsf{X}_2] = \mathsf{X}_3, \qquad [\mathsf{X}_1, \mathsf{X}_3] = \mathsf{X}_4. \tag{5}$$

Obviously, $\{\mathsf{X}_1, \mathsf{X}_3, \mathsf{X}_4\}$ is the Heisenberg–Weyl algebra. We will systematicaly employ this inclusion to save our calculations.

The algebra $\mathfrak{a}$ and respective Lie group—the *shear group* (aka *quartic group*[6] or *Engel group*[7])—is a toy model to try any generalisations[8].[9]

[4]Corwin and Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications. Part I*, 1990.

[5]Kirillov, *Lectures on the Orbit Method*, 2004.

[6]Klink, "Nilpotent Groups and Anharmonic Oscillators", 1994.

[7]Ardentov and Sachkov, "Maxwell Strata and Cut Locus in the Sub-Riemannian Problem on the Engel Group", 2017.

[8]Howe, Ratcliff, and Wildberger, "Symbol Mappings for Certain Nilpotent Groups", 1984.

[9]I. Beltiţă, D. Beltiţă, and Pascu, "Boundedness for Pseudo-Differential Calculus on Nilpotent Lie Groups", 2013.

The shear group

including the Heisenberg group

The corresponding Lie group is a three-step nilpotent $\mathbb{A}$ and the group law is given by:

$$(x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2, \quad x_4 + y_4 + x_1 y_3 + \tfrac{1}{2} x_1^2 y_2), \tag{6}$$

where x_j, $y_j \in \mathbb{R}$ and the *canonical coordinates* are $(x_1, x_2, x_3, x_4) := \exp(x_4 X_4) \exp(x_3 X_3) \exp(x_2 X_2) \exp(x_1 X_1)$. The identity element is $(0, 0, 0, 0)$ and the inverse of an element (x_1, x_2, x_3, x_4) is

$$(-x_1, -x_2, x_1 x_2 - x_3, x_1 x_3 - \frac{1}{2} x_1^2 x_2 - x_4).$$

The group centre is

$$Z(\mathbb{A}) = \{(0, 0, 0, x_4) \in \mathbb{A} : \ x_4 \in \mathbb{R}\}.$$

The Heisenberg group $\mathbb{H}$ is isomorphic to the subgroup

$$\tilde{\mathbb{H}} = \{(x_1, 0, x_3, x_4) \in \mathbb{A} : \ x_j \in \mathbb{R}\} \ \text{ by } \ (x, y, s) \mapsto (x, 0, y, s), \ (x, y, s) \in \mathbb{H}. \tag{7}$$

The shear group and Schrödinger group

The Schrödinger group is the semi-direct product $\mathbb{S} = \mathbb{H} \rtimes_A \mathrm{SL}_2(\mathbb{R})$, where $\mathrm{SL}_2(\mathbb{R})$ is the group of all 2×2 real matrices with the unit determinant. The action A of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is given by:[10]

$$A(g) : (x, y, s) \mapsto (ax + by, cx + dy, s), \tag{8}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $(x, y, s) \in \mathbb{H}$. Let

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},\ x \in \mathbb{R} \right\}$$

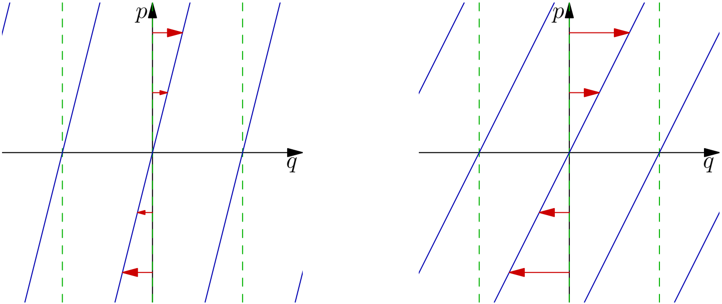
be the subgroup of $\mathrm{SL}_2(\mathbb{R})$, it is easy to check that $\mathbb{A}$ is isomorphic to the subgroup $\mathbb{H} \rtimes_A N$ of $\mathbb{S}$ through the map:

$$(x_1, x_2, x_3, x_4) \mapsto ((x_1, x_3, x_4), n(x_2)) \in \mathbb{H} \rtimes_A N,$$

where $(x_1, x_3, x_4) \in \mathbb{H}$ and $n(x_2) := \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

[10] M. A. d. Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, 2011; Folland, *Harmonic Analysis in Phase Space*, 1989.

The shear group and Schrödinger group



Geometrically: *shear transform* with the angle $\tan^{-1} x_2$:

$$n(x_2)(x_1, x_3) := \begin{pmatrix} 1 & 0 \\ -x_2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 x_1 + x_3 \end{pmatrix}. \tag{9}$$

Physically, for a particle with coordinate x_3 and the constant velocity x_1: after a period of time $-x_2$ the particle will still have the velocity x_1 but its new coordinate will be $x_3 - x_2 x_1$.

UIR of the shear group

induction and the Kirillov's orbit method

Classification of UIRs of the shear group is nicely accomplished by the Kirillov's orbit method (by Kirillov himself[11] :-) and UIRs are explicitly constructed by the Mackey induction procedure.

Avoiding details, the main set of UIRs is parametrised by by two "Planck constants" $\hbar_2$ and $\hbar_4$ and induced by the character $\chi_{\hbar_2\hbar_2}(0, x_2, x_3, x_4) = e^{2\pi(\hbar_2 x_2 + \hbar_4 x_4)}$ of the maximal abelian sungroup $H_a = \{(0, x_2, x_3, x_4)\}$ in $L_2(\mathbb{R})$ is:[12]

$$[\rho_{\hbar_2\hbar_4}(x_1, x_2, x_3, x_4)f](x_1') = e^{2\pi i(\hbar_2 x_2 + \hbar_4(x_4 - x_3 x_1' + \frac{1}{2}x_2 x_1'^2))} f(x_1' - x_1). \quad (10)$$

This representation is irreducible since its restriction to the Heisenberg group $\tilde{\mathbb{H}}$ coincides with the irreducible Schrödinger representation with $\hbar_4$ being the Planck constant.

[11] Kirillov, *Lectures on the Orbit Method*, 2004, § 3.2.

[12] Ibid., 2004, § 3.3, (19).

Coherent state transform

aka voice transform, wavelet transform, etc.

For a G, ρ and a fixed *mother wavelet* $\phi \in H$, the wavelet transform[13] is:

$$[\mathcal{W}^{\rho}_{\phi} f](g) = \left\langle \rho(g^{-1}) f, \phi \right\rangle = \left\langle f, \rho(g)\phi \right\rangle, \qquad g \in G.$$

Let a mother wavelet ϕ be a joint eigenvector of $\rho(h)$ for all $h \in H$:

$$\rho(h)\phi = \chi(h)\phi \quad \text{for all} \quad h \in H. \tag{11}$$

and a character χ of H. Then

$$[\mathcal{W}^{\rho}_{\phi} f](gh) = \bar{\chi}(h)[\mathcal{W}^{\rho}_{\phi} f](g). \tag{12}$$

Thus the restriction of the left regular representation Λ (intertwined with ρ by $\mathcal{W}^{\rho}_{\phi}$) is induced by χ. For a section $s : G/H \to G$, and ϕ satisfying (11), *induced wavelet transform*[14] $\mathcal{W}^{\rho}_{\phi}$ is

$$[\mathcal{W}^{\rho}_{\phi} f](x) = \langle f, \rho(s(x))\phi \rangle, \qquad x \in G/H. \tag{13}$$

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[13] Ali, Antoine, and Gazeau, *Coherent states, wavelets, and their generalizations*, 2014.

[14] Kisil, "Erlangen Programme at Large: an Overview", 2012; Kisil, "Symmetry, Geometry, and Quantization with Hypercomplex Numbers", 2017.

Coherent state transform

for the shear group

For $H = \{(0,0,0,x_4) \in \mathbb{A} : x_4 \in \mathbb{R}\}$ and the character $\chi(0,0,0,x_4) = e^{2\pi i \hbar_4 x_4}$ any function $\phi \in L_2(\mathbb{R})$ satisfies the eigenvector property (11). Thus, for the respective homogeneous space $\mathbb{A}/Z \sim \mathbb{R}^3$ and the section $s : \mathbb{A}/Z \to \mathbb{A}$; $s(x_1,x_2,x_3) = (x_1,x_2,x_3,0)$ the induced wavelet transform is:

$$\begin{aligned}
[\mathcal{W}_\phi f](x_1,x_2,x_3) &= \langle f, \rho_{\hbar_2\hbar_4}(s(x_1,x_2,x_3))\phi\rangle \qquad (14)\\
&= \int_{\mathbb{R}} f(y) e^{-2\pi i(\hbar_2 x_2 + \hbar_4(-x_3 y + \frac{1}{2}x_2 y^2))}\overline{\phi}(y-x_1)\,dy\\
&= e^{-2\pi i \hbar_2 x_2}\int_{\mathbb{R}} f(y) e^{-2\pi i \hbar_4(-x_3 y + \frac{1}{2}x_2 y^2)}\overline{\phi}(y-x_1)\,dy.
\end{aligned}$$

For the Heisenberg group ($x_2 = 0$) it is Fourier–Wigner transform.[15] For the share group it is the *quadratic Fourier transform* or the *Gauss integral transform.*[16]

[15] Folland, *Harmonic Analysis in Phase Space*, 1989.

[16] Neretin, *Lectures on Gaussian Integral Operators and Classical Groups*, 2011; M. A. d. Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, 2011.

CST on the shear group

non-square-integrability, is it an issue?

For a fixed unit vector $\phi \in L_2(\mathbb{R})$, let $L_\phi(\mathbb{A}/Z)$ be the image space of the wavelet transform $\mathcal{W}_\phi$ (14) equipped with the family of inner products parametrised by $x_2 \in \mathbb{R}$

$$\langle u, v\rangle_{x_2} := \int_{\mathbb{R}^2} u(x_1, x_2, x_3)\, \overline{v(x_1, x_2, x_3)}\, \hbar_4 \, dx_1 dx_3 \,. \qquad (15)$$

The respective norm is denoted by $\|u\|_{x_2}$.
$\mathcal{W}_\phi$ is unitary map $H \to L_\phi(\mathbb{A}/Z), \|\cdot\|_{x_2}$, furthermore we have the *orthogonality relation*:

$$\left\langle \mathcal{W}_{\phi_1} f_1, \mathcal{W}_{\phi_2} f_2 \right\rangle_{x_2} = \langle f_1, f_2\rangle\, \overline{\langle \phi_1, \phi_2\rangle} \qquad \text{for any } x_2 \in \mathbb{R}\,. \qquad (16)$$

Then its adjoint:

$$[\mathcal{M}_\phi(x_2) f](t) = \int_{\mathbb{R}^2} f(x_1, x_2, x_3)\rho(x_1, x_2, x_3, 0)\phi(t)\, \hbar_4 \, dx_1 \; dx_3. \qquad (17)$$

is its inverse: $\mathcal{M}_\psi(x_2) \circ \mathcal{W}_\phi = \langle \psi, \phi\rangle\, I$ if $\langle \psi, \phi\rangle = 1$.

Characterisation of CST image

and Lie derivatives

For the right shift $R(g) : f(g') \mapsto f(g'g)$, the covariant transform intertwines $R(g)$ with the action ρ on vacuum states:

$$R(g) \circ \mathcal{W}_\phi = \mathcal{W}_{\rho(g)\phi}. \quad (\text{cf. } \Lambda(g) \circ \mathcal{W}_\phi = \mathcal{W}_\phi \circ \rho(g)). \tag{18}$$

Let ρ be a UIR of G, which can be extended[17] by integration to a vector space V of functions or distributions on G. If $\phi \in H$ satisfy the

$$\rho(a)\phi = 0, \quad \text{where } \rho(a)\phi = \int_G a(g)\, \rho(g)\phi \, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{v}(g) = \langle v, \rho(g)\phi \rangle$ obeys the condition:[18]

$$R(\bar{a})v = 0, \qquad \text{where} \quad R(\bar{a}) = \int_G \bar{a}(g)\, R(g)\, dg\,, \tag{19}$$

with R being the right regular representation of G.

[17] Feichtinger and Gröchenig, "Banach spaces related to integrable group representations and their atomic decompositions. I", 1989.

[18] Kisil, "Erlangen Programme at Large: an Overview", 2012; Kisil, "The Real and Complex Techniques in Harmonic Analysis from the Point of View of Covariant Transform", 2014.

The shear group CST

image characterisation: structural condition

The derived representations of the basis $\{X_1, X_2, X_3, X_4\}$ of $\mathfrak{a}$ are:

$$d\rho^{X_1}_{\hbar_2\hbar_4} = -\frac{d}{dy}; \quad d\rho^{X_2}_{\hbar_2\hbar_4} = 2\pi i\hbar_2 + \pi i\hbar_4 y^2; \quad d\rho^{X_3}_{\hbar_2\hbar_4} = -2\pi i\hbar_4 y; \tag{20}$$

Lie derivative $\mathfrak{L}^X$ is the derived right regular representation:

$$\mathfrak{L}^{X_1} = \partial_1; \quad \mathfrak{L}^{X_2} = \partial_2 + x_1\partial_3 - i\pi\hbar_4 x_1^2 I; \quad \mathfrak{L}^{X_3} = \partial_3 - 2\pi i\hbar_4 x_1 I; \tag{21}$$

Any function ϕ satisfies the relation for the derived representation

$$\left((d\rho^{X_3}_{\hbar_2\hbar_4})^2 - 4\pi i\hbar_4\, d\rho^{X_2}_{\hbar_2\hbar_4} - 8\pi^2\hbar_2\hbar_4 I\right)\phi = 0.$$

The image $f \in L_\phi(\mathbb{A}/Z)$ of the wavelet transform $\mathcal{W}_\phi$ is annihilated by the respective Lie derivatives operator $\mathcal{S}f = 0$ where

$$\begin{aligned} \mathcal{S} &= (\mathfrak{L}^{X_3})^2 + 4\pi i\hbar_4\mathfrak{L}^{X_2} - 8\pi^2\hbar_2\hbar_4 I \\ &= \partial_{33}^2 + 4\pi i\hbar_4\partial_2 - 8\pi^2\hbar_2\hbar_4 I\,. \end{aligned} \tag{22}$$

This will be called the *structural condition* because it is determined by the structure of the particular representation $\rho_{\hbar_2\hbar_4}$ (10).

A particular choice of a mother wavelet ϕ such that ϕ lies in $L_2(\mathbb{R})$ and is a null solution to the "first order" operator, cf. (20):

$$d\rho_{\hbar_2\hbar_4}^{X_1+aX_2+iEX_3} = d\rho_{\hbar_2\hbar_4}^{X_1} + a\, d\rho_{\hbar_2\hbar_4}^{X_2} + iE\, d\rho_{\hbar_2\hbar_4}^{X_3}, \tag{23}$$

where a and E some real constants. It is clear that, the function

$$\phi(y) = \exp\left(\frac{\pi i a \hbar_4}{3} y^3 + \pi E \hbar_4 y^2 + 2\pi i a \hbar_2 y\right),$$

is a generic solution and square integrability of ϕ requires that $E\hbar_4$ is strictly negative. Furthermore, for the purpose of this work it is sufficient to use the simpler mother wavelet corresponding to the value $a = 0$:

$$\phi(y) = e^{\pi E \hbar_4 y^2}, \qquad \hbar_4 > 0,\ E < 0. \tag{24}$$

If $a \neq 0$ then we obtain Airy beam decomposition (cubic Fourier transform).[19]

[19] Torre, "A Note on the Airy Beams in the Light of the Symmetry Algebra Based Approach", 2009.

The shear group CST

image characterisation: the analytic condition

Any function f in $L_\phi(\mathbb{A}/Z)$ for ϕ (24) satisfies $\mathcal{C}f = 0$ for the partial differential operator produced from (23) with $a = 0$:

$$\mathcal{C} = \left(\mathfrak{L}^{X_1} - \mathrm{i}E\mathfrak{L}^{X_3}\right) = \partial_1 - \mathrm{i}E\partial_3 - 2\pi\hbar_4 E x_1 \,. \tag{25}$$

By *peeling* (multiplication with a suitable factor) it can be converted into the Cauchy–Riemann equation, we call (25) the *analyticity condition*.
The CST (14) with the Gaussian for the Haisenberg group ($x_2 = 0$) becomes the Fock–Segal–Bargmann transform to analytic function of $z = -Ex_1 + \mathrm{i}x_3$.
Since the structural operator $\mathcal{S} = \partial_{33}^2 + 4\pi\mathrm{i}\hbar_4\partial_2 - 8\pi^2\hbar_2\hbar_4 I$ (22) is a Schrödinger equation of a free particle (with x_2 being time) we get
Physical characterisation of $L_\phi(\mathbb{A}/Z)$: consists of wavefunctions expanded from the phase space $\mathbb{R}^2$ to $\mathbb{R}^2 \times \mathbb{R}$ by free time-evolution.

Geometric dynamics of HO

from the Heisenberg group

The harmonic oscillator with mass m and frequency ω is quantised

$$\begin{aligned} H &= \frac{1}{2m}(\mathrm{i}d\tilde{\sigma}_h^{X_1})^2 + \frac{m\omega^2}{2}(\mathrm{i}d\tilde{\sigma}_h^{x_3})^2 \qquad (26)\\ &= -\frac{1}{2m}\partial_{11}^2 - \frac{m\omega^2}{2}\partial_{22}^2 + \frac{2\pi \mathrm{i}h}{m}x_3\partial_1 + \frac{2\pi^2h^2}{m}x_3^2 \end{aligned}$$

Our aim is dynamics in geometric terms by lowering the order of the differential operator (26) using the analyticity condition:

$$\tilde{H} = H + (A\partial_1 + \mathrm{i}B\partial_3 + CI)(\mathfrak{L}^{X_1} - \mathrm{i}E\mathfrak{L}^{x_3})$$

This requires $A = \frac{1}{2m}$, $B = -\frac{1}{2}\omega$, $C = -\pi h\omega x_1$ and $E = -m\omega$.

$$\tilde{H} = \frac{2\pi \mathrm{i}h}{m}x_3\partial_1 - 2\pi \mathrm{i}hm\omega^2x_1\partial_3 + \left(\pi h\omega + \frac{2\pi^2h^2}{m}(x_3^2 - m^2\omega^2x_1^2)\right)I\,. \qquad (27)$$

The solution is given by the rotation of the complex variable $x_3 - \mathrm{i}m\omega x_1$. *New observation:* the value of E is uniquely defined and the corresponding vacuum vector $\phi(q) = e^{\pi hEq^2} = e^{-\pi hm\omega q^2}$ is fixed.

Geometric dynamics of HO

from the shear group

Similarly, the Hamiltonian of HO for the shear group ($x_2 \neq 0$) is

$$\begin{aligned} H &= \left(\frac{1}{2m}(id\tilde{\rho}^{X_1}_{\hbar_4})^2 + \frac{m\omega^2}{2}(id\tilde{\rho}^{X_3}_{\hbar_4})^2\right) \\ &= -\frac{1}{2m}\partial^2_{11} - \frac{1}{2m}x_2^2\partial^2_{33} - \frac{m\omega^2}{2}\partial^2_{33} - \frac{1}{m}x_2\partial^2_{13} \\ &\quad + \frac{2\pi i\hbar_4}{m}x_3\partial_1 + \frac{2\pi i\hbar_4}{m}x_2x_3\partial_3 - \frac{1}{m}(-\pi i\hbar_4 x_2 - 2\pi^2\hbar_4^2x_3^2)I \end{aligned} \tag{28}$$

Adjusting by the analytic (25) and structural (22) conditions:

$$H_1 = H + (A\partial_1 + B\partial_2 + C\partial_3 + KI)\mathcal{C} + F\mathcal{S}\,.$$

To eliminate all second order derivatives take $A = \frac{1}{2m}$, $B = 0$, $C = \frac{1}{m}(\frac{i}{2}E + x_2)$, $K = \frac{\pi\hbar_4}{m}x_1(E + 2ix_2)$. and $F = -\frac{1}{2m}(ix_2 - E)^2 + \frac{m\omega^2}{2}$.
Difference with the Heisenberg group: there is no restrictions for the parameter E, any squeezed states[20]-[21] $e^{\pi\hbar_4 E y^2}$, $E < 0$ can be mother wavelet.

[20] Gazeau, *Coherent States in Quantum Physics*, 2009.

[21] M. A. d. Gosson, *Symplectic Methods in Harmonic Analysis and in Mathematical Physics*, 2011; M. A. d. Gosson, "Quantum Blobs", 2013.

Geometric dynamics of HO on the shear group solution

The adjusted Hailtonian is:

$$H_1 = \frac{2\pi i\hbar_4}{m}\left((x_3 + x_1x_2)\partial_1 - \left((ix_2 - E)^2 - m^2\omega^2\right)\partial_2 - (E^2x_1 - x_2x_3)\partial_3\right)$$
$$- \frac{\pi\hbar_4}{m}\left(8i\pi\hbar_2 E x_2 - ix_2 + 4\pi\hbar_2 x_2^2 - 2\pi\hbar_4 x_3^2\right.$$
$$\left. + 4\pi\hbar_2 m^2\omega^2 + E - 4\pi\hbar_2 E^2 + 4i\pi E\hbar_4 x_1^2 x_2 + 2\pi\hbar_4 E^2 x_1^2\right) I\,.$$

Using the analyticity condition in the variable $z = x_3 + iEx_1$:

$$f(t, x_1, x_2, x_3) = \frac{\sqrt{E + m\omega}}{\sqrt{ix_2 + E + m\omega}} \tag{29}$$
$$\times \exp\left(i\pi\omega t - \pi\hbar_4 E x_1^2 - 2\pi i\hbar_2 x_2 - \pi\hbar_4 \frac{(x_3 - iEx_1)^2}{ix_2 + E + m\omega}\right)$$
$$\times f_1\left(e^{2\pi i\omega t}\frac{x_3 - iEx_1}{ix_2 + E + m\omega}, e^{4\pi i\omega t}\frac{m\omega - (ix_2 + E)}{m\omega + (ix_2 + E)}\right).$$

where $f_1(z, u)$ is an arbitrary function of two variables such that:

- analytic in the first variable.
- solves heat-like eqn. $\partial_u f_1(z, u) = -\frac{1}{8\pi\hbar_4 m\omega}\partial^2_{zz} f_1(z, u)$

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The solution: bounds for a possible squeeze

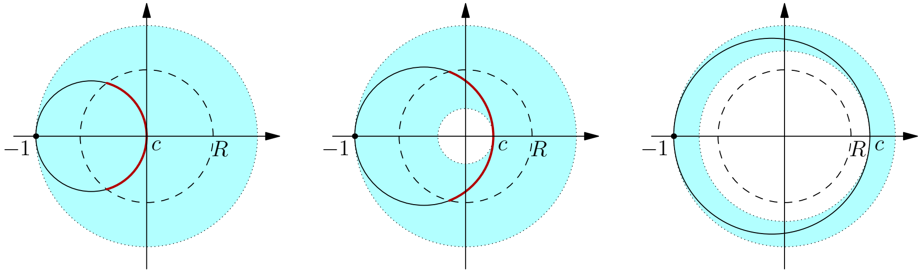

Figure: Shear parameter and analytic continuation. The solid circle is the image of the line $ix_2 + E$ under the Cayley transformation. The shadowed region (the annulus with radii c and 1) is obtained from the solid circle under rotation around the origin. The dashed circle of the radius R bounds the domain of the analytic continuation of the heat equation solution.
Left: $E = m\omega$ (thus $c = 0$)—there always exists a part of the shaded region inside the circle of a radius R (even for $R = 0$).
Middle: some E within the bound—there is a thick arc inside of the dashed circle, the arc corresponds to values of x_2 with a meaningful solution (29).
Right: a shear parameter E is outside of the range, a state is squeezed too much, no values of x_2 are allowed in (29).

Bibliography I

S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. *Coherent states, wavelets, and their generalizations*. Second. Theoretical and Mathematical Physics. Springer, New York, 2014, pp. xviii+577. URL: http://dx.doi.org/10.1007/978-1-4614-8535-3.

F. Almalki and V. V. Kisil. "Geometric Dynamics of a Harmonic Oscillator, Non-Admissible Mother Wavelets and Squeezed States". In: (2018). E-print: arXiv:1805.01399, p. 23.

A. A. Ardentov and Y. L. Sachkov. "Maxwell Strata and Cut Locus in the Sub-Riemannian Problem on the Engel Group". In: *Regul. Chaotic Dyn.* 22.8 (2017), pp. 909–936. URL: https://doi.org/10.1134/S1560354717080020.

I. Beltiţă, D. Beltiţă, and M. Pascu. "Boundedness for Pseudo-Differential Calculus on Nilpotent Lie Groups". In: *Geometric methods in physics*. Trends Math. Birkhäuser/Springer, Basel, 2013, pp. 87–97.

Bibliography II

L. J. Corwin and F. P. Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications. Part I.* Vol. 18. Cambridge Studies in Advanced Mathematics. Basic theory and examples. Cambridge University Press, Cambridge, 1990, pp. viii+269.

H. G. Feichtinger and K. H. Gröchenig. "Banach spaces related to integrable group representations and their atomic decompositions. I". In: *J. Funct. Anal.* 86.2 (1989), pp. 307–340. URL: http://dx.doi.org/10.1016/0022-1236(89)90055-4.

G. B. Folland. *Harmonic Analysis in Phase Space.* Vol. 122. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 1989, pp. x+277.

J.-P. Gazeau. *Coherent States in Quantum Physics.* Wiley-VCH Verlag, 2009.

Bibliography III

M. de Gosson. *Symplectic Geometry and Quantum Mechanics.* Vol. 166. Operator Theory: Advances and Applications. Advances in Partial Differential Equations (Basel). Basel: Birkhäuser Verlag, 2006, pp. xx+367.

M. A. de Gosson. *Symplectic Methods in Harmonic Analysis and in Mathematical Physics.* Vol. 7. Pseudo-Differential Operators. Theory and Applications. Birkhäuser/Springer Basel AG, Basel, 2011, pp. xxiv+337. URL: http://dx.doi.org/10.1007/978-3-7643-9992-4.

M. A. de Gosson. "Quantum Blobs". In: *Found. Phys.* 43.4 (2013), pp. 440–457. URL: https://doi.org/10.1007/s10701-012-9636-x.

UNIVERSITY OF LEEDS

Bibliography IV

R. Howe, G. Ratcliff, and N. Wildberger. "Symbol Mappings for Certain Nilpotent Groups". In: *Lie group representations, III (College Park, Md., 1982/1983)*. Vol. 1077. Lecture Notes in Math. Berlin: Springer, 1984, pp. 288–320. URL: http://dx.doi.org/10.1007/BFb0072342.

A. A. Kirillov. *Lectures on the Orbit Method*. Vol. 64. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2004, pp. xx+408.

V. V. Kisil. "Erlangen Programme at Large: an Overview". In: *Advances in Applied Analysis*. Ed. by S.V. Rogosin and A.A. Koroleva. E-print: arXiv:1106.1686. Basel: Birkhäuser Verlag, 2012. Chap. 1, pp. 1–94.

V. V. Kisil. "The Real and Complex Techniques in Harmonic Analysis from the Point of View of Covariant Transform". In: *Eurasian Math. J.* 5 (2014). E-print: arXiv:1209.5072. On-line, pp. 95–121.

Bibliography V

V. V. Kisil. "Symmetry, Geometry, and Quantization with Hypercomplex Numbers". In: *Geometry, Integrability and Quantization* 18 (2017). E-print: arXiv:1611.05650, pp. 11–76.

W. H. Klink. "Nilpotent Groups and Anharmonic Oscillators". In: *Noncompact Lie groups and some of their applications (San Antonio, TX, 1993)*. Vol. 429. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, 1994, pp. 301–313.

Y. A. Neretin. *Lectures on Gaussian Integral Operators and Classical Groups*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011, pp. xii+559. URL: http://dx.doi.org/10.4171/080.

A. Torre. "A Note on the Airy Beams in the Light of the Symmetry Algebra Based Approach". In: *Journal of Optics A: Pure and Applied Optics* 11.12 (2009), p. 125701. URL: http://stacks.iop.org/1464-4258/11/i=12/a=125701.